Chaoticity and Coherence in Bose-Einstein Condensation and Correlations*

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Abstract

We review the properties of chaoticity and coherence in Bose-Einstein condensation and correlations, for a dense boson system in its mean-field represented approximately by a harmonic oscillator potential. The order parameter and the nature of the phase transition from the chaotic to the condensate states are studied for different fixed numbers of bosons. The two-particle correlation function in momentum space is calculated to investigate how the Bose-Einstein correlation depends on the degree of condensation and other momentum variables. We generalize the Bose-Einstein correlation analysis to three-particle correlations to show its dependence on the degree of condensation.

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1 Introduction

As is well known, a fundamental assumption for the occurrence of Bose-Einstein correlation (BEC) is the presence of a chaotic source of identical bosons [1, 2]. The Bose-Einstein correlation occurs in a chaotic source but not in a coherent source [3, 4, 5, 6, 7, 8].

The properties of chaoticity and coherence are complementary attributes. Both chaoticity and coherence should be examined on equal footings in a single theoretical framework with the description of both the BE condensation and BE correlations. In such a unified framework, it is then possible to investigate not only the states of chaoticity and coherence, but also the transition from a chaotic state to a coherent state. How can the degrees of chaoticity or coherence be quantified? Is the transition from a chaotic state to a coherent state a first-order with a sudden onset, or is it a gradual transition that is closer to a second-order? What is the relevant order parameter that best describes the transition? How does Bose-Einstein condensation quantitatively affects the two-particle and three-particle Bose-Einstein correlations?

Questions of Bose-Einstein correlations and condensation arise not only in atomic physics [9, 10, 11] but also in high-energy heavy-ion collisions [5, 6, 8] where pions are the most copiously produced particles. The use of two-pion Bose-Einstein correlations to probe the source coherence was proposed at the end of 1970s [12, 3]. The introduction of the "chaoticity" parameter λ of BEC in pions is only a tool to represent experimental data. However, the experimental measurement of λ is beset by the presence of many other effects such as particle misidentification, long-live resonance decay, final state Coulomb interaction, non-Gaussian source distribution, etc. [5, 7]. The explanation of the experimental λ results remains an open question. In 1993, S. Pratt proposed a pion laser model in high energy collisions and studied the influence of pion laser on two-pion Bose-Einstein correlation function and the chaoticity parameter [13]. In 1998, T. Csörgő and J. Zimányi investigated the effect of Bose-Einstein condensation on two-pion Bose-Einstein correlations [14]. They utilized Gaussian formulas describing the space and momentum distributions of a static non-relativistic boson system, and investigated the influence of the condensation on pion multiplicity distribution. In 2007, C. Y. Wong and W. N. Zhang studied how λ in Bose-Einstein correlations depends on the degree of Bose-Einstein condensation or chaoticity, for static non-relativistic and relativistic boson gases within a spherical mean-field harmonic oscillator potential [15]. The model can be analytically solved in the non-relativistic case and be used in atomic physics [9, 10, 11]. The limiting conditions and circumstances under which the parameter λ can be approximately related to the degrees of chaoticity were clarified [15]. A similar study for cylindrical static boson gas sources was completed [16] and the chaoticity parameter λ in two-pion Bose-Einstein correlations in an expanding boson gas model was recently examined [17]. The investigation of chaoticity and coherence was also carried out using a model of q-deformed oscillator algebraic commutative relations [18] and the model of partial indistinguishability and coherence of closely located emitters [19]. In another related topic, initial conditions such as the color-glass condensate (CGC) with the coherent production of partons [20] in heavy-ion collisions may also lead to condensate formation [21].

Recently, experimental investigation of the source coherence in Pb-Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV at the Large Hadron Collider (LHC) was carried out by the ALICE collaboration [22]. A substantial degree of source coherence was measured [22] using a new three-pion Bose-Einstein correlations technique with an improvement over past efforts [23, 24, 25, 26]. Earlier work on three-particle correlations were carried out in [6, 13, 27, 28, 29, 30, 31, 32, 33, 34].

A proper theoretical framework to study the above topics is the theory of the Bose-Einstein condensation and correlations in their own mean field potential [15]. We would like to review the essential elements here and examine further the related question of three-body correlations.

2 Bose-Einstein Condensation for attractively Interacting Bosons

We seek a description of chaoticity in Bose-Einstein correlations through the consideration of Bose-Einstein condensation. Why is Bose-Einstein condensation relevant to Bose-Einstein correlations (BEC)? Glauber in many private communications and in his talk in QM2005 suggested that the consistent experimental observations of $\lambda < 1$ may be due partly to the coherence of the pions in Bose Einstein correlations [35]. Furthermore, there have been major advances in Bose-Einstein condensation in atomic physics [9, 10, 11]. In particular, the works of Politzer [9], and Naraschewski & Glauber [10] reveals that BE condensation and the BE correlations are intimately related.

We envisage the possibility of the occurrence of a Bose-Einstein condensation in dense boson media of identical bosons with the following reasoning [15, 16, 17]

1. Identical bosons with mutual attractive interaction generate a mean field potential, which depends on the boson density $\rho(r)$ as [36]

$$V(r) = -4\pi f(0)\rho(r) \sim \frac{1}{2}\hbar\omega \left(\frac{r}{a}\right)^2, \tag{1}$$

   where $f(0)$ is the forward scattering amplitude, and a is the length scale that defines the spatial region of boson occupation.

2. Therefore, for a given length scale a, the $\hbar\omega$ of the underlying mean-field potential increases with increasing density ρ of the produced bosons.

3. The order parameter that determines the degree of BE coherence or chaoticity is $T/\hbar\omega$. Thus the order parameter $T/\hbar\omega$ decreases with increasing boson density.

4. For a given temperature T at freeze out, a high density of produced bosons will lead to a lower value of the order parameter $T/\hbar\omega$, which in turn will lead to a greater condensate fraction $f_0{=}N_0/N$, where N is the total number of bosons and N_0 is the number of bosons in the lowest state. A greater condensate fraction f_0 brings about a greater coherence in Bose-Einstein correlations and a reduction in the degree of chaoticity.

In high energy heavy-ion collisions when bosons (gluons or pions) are copiously produced within a small region in a short time interval, the density of the bosons increases as the collision energy increases. Following the above reasoning, generalized to systems with differential transverse and longitudinal spatial distributions, we expect the occurrence of boson condensation in high energy heavy-ion collisions at some high collision energies. It is useful to examine the Bose-Einstein condensation for bosons in an exactly solvable model.

3 Bose-Einstein Condensation for Bosons in a Spherical Harmonic Oscillator Potential

We consider a system of N bosons in a spherical harmonic oscillator potential, which arises either externally or from the bosons' own mean-fields. We study how the occupation numbers of different states change as a function of the temperature T, in relation to the oscillator frequency $\hbar\omega$. Bose-Einstein condensation occurs when the occupation number N_0 for the lowest state (the condensate state) is a substantial fraction of the total particle number N. The degree of coherence or chaoticity is quantified by the condensate fraction $f_0 = N_0/N$, which varies as a function of the order parameter $T/\hbar\omega$.

In such a study, it is important to use the proper statistical ensemble [9]. In a grand canonical ensemble, we fix the chemical potential μ and the temperature T, and we allow the number of particles N_n in the n-th single-particle state to vary. We obtain the average occupation number for the single-particle state n to be $N_n = \langle a_n^+ a_n \rangle$. The square fluctuation of N_n is then given by

$$\langle (a_n^+ a_n - \langle a_n^+ a_n \rangle)^2 \rangle \approx N_n(N_n + 1). \tag{2}$$

As the fluctuation of N_n in a grand canonical ensemble is of the same order as the occupation number itself, we cannot treat the lowest $n = 0$ state in the grand canonical ensemble. The lowest $n = 0$ state needs to be treated in the canonical ensemble with a fixed total number of bosons.

It was shown however that while the lowest $n = 0$ state needs to be treated in the canonical ensemble, the $n > 0$ state can be treated in the grand canonical ensemble without incurring large errors [9]. We shall follow such a description for the ensemble of N identical bosons in a spherical harmonic oscillator potential. In such a canonical ensemble for the lowest n=0 state but a grand canonical ensemble for the n>0 states, the total number of bosons is fixed and yields the condensate number condition

$$N = N_0 + \sum_{n=1,2,3,...} N_n = \frac{z}{1-z} + \sum_{n=1,2,3,...} \frac{g_n z e^{-(\epsilon_n - \epsilon_0)/T}}{1 - z e^{-(\epsilon_n - \epsilon_0)/T}}, \tag{3}$$

where $z = e^{\mu/T}$ is the fugacity of the system, g_n is the degeneracy number g_n=$(n$+1$)(n$+2$)$/2 for the n-th single-particle level, and ϵ_n is the single-particle energy in the spherical harmonic oscillator potential

$$\epsilon_n = (n + 3/2)\hbar\omega. \tag{4}$$

For a given N, equation (3) contains only a single unknown, z, which can be solved as a function of the order parameter $T/\hbar\omega$. The solutions of z for N=25, 500, 1000 and 2000 are given in Fig. 1, and the corresponding condensate fractions $f_0 = N_0/N$ are shown in Fig. 2.

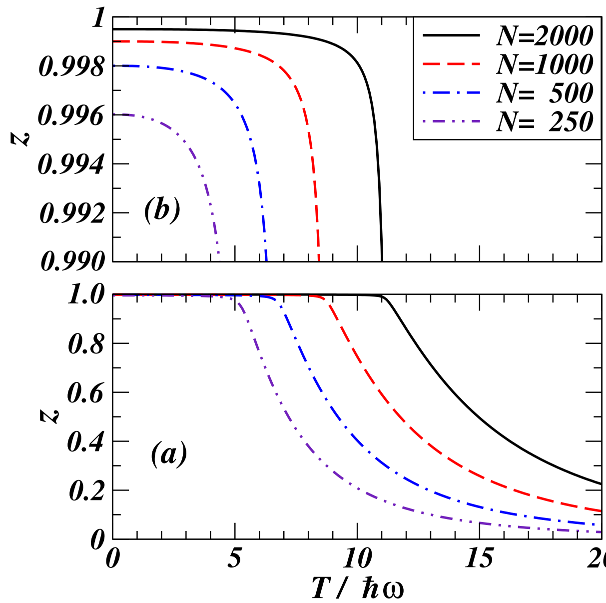

Figure 1: (Color online) (a) The fugacity parameter z satisfying the condensate number condition Eq. (3) for different boson numbers N in a spherical harmonic oscillator potential, as a function of the order parameter $T/\hbar\omega$ and (b) an expanded view in the $z \sim 1$ region.

We observe in Fig. 1 that the fugacity parameter z is close to unity in the strongly coherent region at low temperatures. In fact, the fugacity parameter z at T=0 assumes the value $z(T{=}0) = N/(N+1)$. For a given boson number N, the fugacity z decreases very slowly in the form of a plateau, as the temperature increases from $T = 0$. The plateau region persists until the condensate temperature T_c is reached, and z then decreases rapidly thereafter. The greater the number of bosons N, the greater is the plateau region, as shown in Fig. 1(b). For example, for $N = 2000$ the value of z is close to unity for $0 < T/\hbar\omega < 11$ in the plateau,.

We note in Fig. 2 that for a given value of the total number of bosons N in the spherical harmonic oscillator potential, the condensate fraction f_0 is close to unity when the order parameter $T/\hbar\omega$ is below a limit, and this limit depends on N. We can plot the condensate fraction f_0 as a function of the order parameter $T/\hbar\omega$. The functional form of $f_0(T)$ can be approximated by

$$f_0(T) = \begin{cases} 1 - [(T/\hbar\omega)/(T_c/\hbar\omega)]^3 & \text{for } (T/\hbar\omega) \le (T_c/\hbar\omega), \\ O(1/N) \to 0 & \text{for } (T/\hbar\omega) \ge (T_c/\hbar\omega). \end{cases} \tag{5}$$

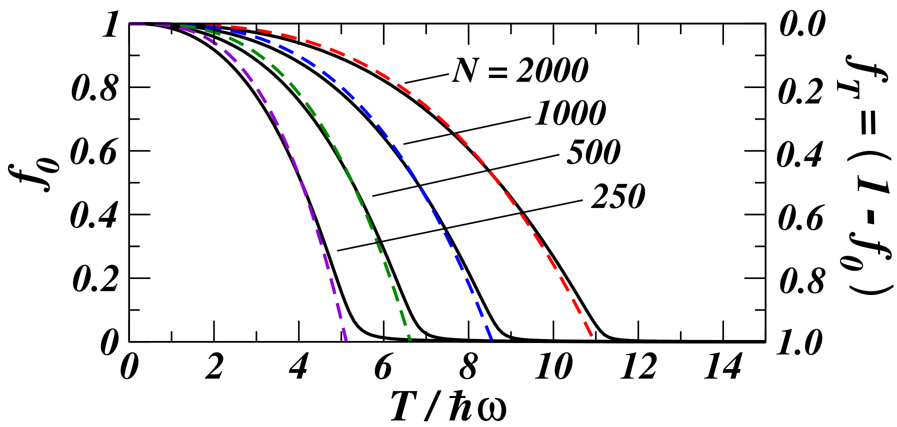

Figure 2: (Color online) Solid curves represent the condensate fractions $f_0(T)$, calculated with the condensate number condition Eq. (3), as a function of $T/\hbar\omega$ for different boson numbers N in a spherical harmonic oscillator potential. The abscissa labels for the corresponding chaotic fraction $f_T(T)$=$[1 - f_0(T)]$ are indicated on the right. The dashed curves are the fits to the solid curve results of $f_0(T)$ with the function $1-[(T/\hbar\omega)/(T_c/\hbar\omega)]^3$ of Eq. (5) where the values of $T_c/\hbar\omega$ for different N values are listed in Table I.

The results from the above one-parameter fit to $f_0(T)$ are shown as the dashed curves in Fig. 2, to be compared with the $f_0(T)$ calculated with the condensate configuration condition Eq. (3) shown as the solid curves. The values of $T_c/\hbar\omega$ that give the best fit to $f_0(T)$ for different N values are listed in Table I.

The above results provide a comprehensive description for the transition from a chaotic state to a coherent state. Fig. 2 indicates that the transition from the completely chaotic state with f_0=0 to the state of coherence with $f_0 \to 1$ is a second-order-type transition under a gradual decrease of the order parameter $T/\hbar\omega$. It is not a first-order phase transition.

Table I. Critical order parameter $T_c/\hbar\omega$ obtained from (i) fitting f_0 as a function of $T/\hbar\omega$ with Eq. (5), and from (ii) the analytical formula of Eq. (6).

Number of Bosons N	$T_c/\hbar\omega$ obtained from fitting f_0 with Eq. (5)	$T_c/\hbar\omega$ obtained with Eq.(6)
2000	10.97	11.00
1000	8.56	8.53
500	6.63	6.62
250	5.12	5.13

It is remarkable that the critical order parameter $T_c/\hbar\omega$ and the boson number N obeys the following simple relationship

$$T_c/\hbar\omega \quad = \quad 0.6777 N^{0.36666}, \tag{6}$$

as shown by the third column in Table I. Thus, the knowledge of N suffices to determine the critical order parameter $T_c/\hbar\omega$ by the above simple equation and the knowledge of $T_c/\hbar\omega$ subsequently yields the approximate condensate fraction at all other temperatures by Eq. (5).

4 Single-particle and Two-Particle Density Matrices in Momentum Space

The determination of the fugacity z from the condensate number condition (3) allows the calculation of various physical quantities. Specifically, the one-body density matrix in momentum space is given by

$$G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_1') = \sum_{n=0}^{\infty} u_n^*(\boldsymbol{p}_1') u_n(\boldsymbol{p}_1) \langle \hat{a}_n^\dagger \hat{a}_n \rangle, \tag{7}$$

where $u_n(\boldsymbol{p})$ is the single-particle wave function and the occupation number $N_n{=}\langle \hat{a}_n^\dagger \hat{a}_n \rangle$ can be inferred from the terms in the summation in Eq. (3). The two-particle density matrix in momentum space

$$G^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2') = \sum_{\rm klmn} u_{\rm k}^*(\boldsymbol{p}_1') u_{\rm l}^*(\boldsymbol{p}_2') u_{\rm m}(\boldsymbol{p}_2) u_n(\boldsymbol{p}_1) \langle \hat{a}_{\rm k}^\dagger \hat{a}_{\rm l}^\dagger \hat{a}_{\rm m} \hat{a}_{\rm n} \rangle \tag{8}$$

can be written in terms of one-body density matrices as [10, 15]

$$\begin{aligned} G^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2') &= G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_1') G^{(1)}(\boldsymbol{p}_2, \boldsymbol{p}_2') + G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_2') G^{(1)}(\boldsymbol{p}_2, \boldsymbol{p}_1') \\ &+ \sum_{n=0}^{\infty} u_n^*(\boldsymbol{p}_1') u_n^*(\boldsymbol{p}_2') u_n(\boldsymbol{p}_2) u_n(\boldsymbol{p}_1) \left\{ \langle \hat{a}_n^\dagger \hat{a}_n \hat{a}_n \rangle - 2 \langle \hat{a}_n^\dagger \hat{a}_n \rangle \langle \hat{a}_n^\dagger \hat{a}_n \rangle \right\}. \end{aligned} \tag{9}$$

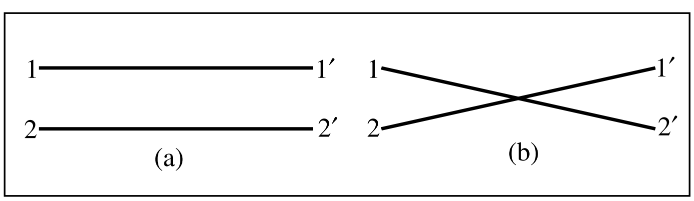

Figure 3: Two-particle distribution function expanded in terms of products of one-particle distribution functions in uncorrelated mean-field approximation.

The uncorrelated part in the first two terms of the above two-particle density matrix, $G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_1') G^{(1)}(\boldsymbol{p}_2, \boldsymbol{p}_2') + G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_2') G^{(1)}(\boldsymbol{p}_2, \boldsymbol{p}_1')$, is represented schematically by the diagram in Fig. 3. Our task is to obtain the correlated part arising from Bose-Einstein condensation represented by the last term in Eq. (9).

In the limit of a large number of bosons N in a grand canonical ensemble for the non-condensed states, the contributions from the set of $\{n > 0\}$ states in the

summation in Eq. (9) can be neglected. We are left with only the $n = 0$ condensate state contribution for this summation.

To describe the contribution from the $n = 0$ condensate state, we shall follow Ref. [9, 10] and use the canonical ensemble which gives the canonical fluctuation

$$\langle(\hat{a}_0^\dagger \hat{a}_0 - \langle \hat{a}_0^\dagger \hat{a}_0 \rangle)^2\rangle = \langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle - \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \langle \hat{a}_0^\dagger \hat{a}_0 \rangle = O(N_0). \tag{10}$$

Thus, we have

$$\langle \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0 \rangle - 2\langle \hat{a}_0^\dagger \hat{a}_0 \rangle \langle \hat{a}_0^\dagger \hat{a}_0 \rangle = -\langle \hat{a}_0^\dagger \hat{a}_0 \rangle \langle \hat{a}_0^\dagger \hat{a}_0 \rangle + O(N_0). \tag{11}$$

In the limit of a large number of particles, we can neglect the last term $O(N_0)$ in the above equation which is small in comparison with the first term of order N_0^2. The two-particle distribution of Eq. (9) is therefore

$$\begin{aligned} G^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1, \boldsymbol{p}_2) = G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_1) G^{(1)}(\boldsymbol{p}_2, \boldsymbol{p}_2) + |G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_2)|^2 \\ - N_0^2 |u_0(\boldsymbol{p}_1)|^2 |u_0(\boldsymbol{p}_2)|^2, \end{aligned} \tag{12}$$

which gives the conditional probability for the occurrence of a pion of momentum $\boldsymbol{p}_1$ in coincidence with another identical pion of momentum $\boldsymbol{p}_2$.

5 Two-Particle Momentum Correlation Function

In BE correlation measurements, we normalize the probability relative to the probability of detecting particle $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, and define the momentum correlation function $C(\boldsymbol{p}_1, \boldsymbol{p}_2)$ as

$$C(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{G^{(2)}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1, \boldsymbol{p}_2)}{G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_1) G^{(1)}(\boldsymbol{p}_2, \boldsymbol{p}_2)}. \tag{13}$$

It is convenient to introduce the average and the relative momenta of the pair

$$\boldsymbol{p} = (\boldsymbol{p}_1 + \boldsymbol{p}_2)/2, \qquad \boldsymbol{q} = \boldsymbol{p}_1 - \boldsymbol{p}_2. \tag{14}$$

The momentum correlation function can be expressed as a function of the kinematic variables $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ or alternatively of $\boldsymbol{p}$ and $\boldsymbol{q}$. From Eq. (12), we have the general expression for the correlation function

$$C(\boldsymbol{p}, \boldsymbol{q}) = C(\boldsymbol{p}_1, \boldsymbol{p}_2) = 1 + \frac{|G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_2)|^2 - N_0^2 |u_0(\boldsymbol{p}_1)|^2 |u_0(\boldsymbol{p}_2)|^2}{G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_1) G^{(1)}(\boldsymbol{p}_2, \boldsymbol{p}_2)}. \tag{15}$$

This is the general Bose-Einstein correlation function for all situations: coherent, chaotic, and the transition between coherent and chaotic systems.

The evaluation of the correlation function $C(\boldsymbol{p}, \boldsymbol{q})$ in Eq. (15) requires the knowledge of $G^{(1)}(\boldsymbol{p}_1, \boldsymbol{p}_2)$ and the ground state wave function $u_0(\boldsymbol{p}_1)$. For a system of bosons in a spherical harmonic oscillator, the wave functions are all known, and the correlation function can be written out analytically. Specifically, we have

$$G^{(1)}(\boldsymbol{p}_1,\boldsymbol{p}_2) = \sum_{k=1}^{\infty} z^k \tilde{G}_0(\boldsymbol{p}_1,\boldsymbol{p}_2;k\beta\hbar\omega), \tag{16}$$

$$\tilde{G}_0(\boldsymbol{p}_1,\boldsymbol{p}_2;\tau)=\left(\frac{a^2}{\pi\hbar^2(1-e^{-2\tau})}\right)^{3/2}\exp\left(-\frac{a^2}{\hbar^2}\frac{(\boldsymbol{p}_1^2+\boldsymbol{p}_2^2)(\cosh\tau-1)+(\boldsymbol{p}_1-\boldsymbol{p}_2)^2}{2\sinh\tau}\right), \tag{17}$$

and the ground state wave function is

$$u_0(\boldsymbol{p}) = \left(\frac{a^2}{\pi\hbar^2}\right)^{3/4} \exp\left\{-\frac{a^2}{\hbar^2}\frac{\boldsymbol{p}^2}{2}\right\}. \tag{18}$$

The knowledge of the single-particle $G^{(1)}(\boldsymbol{p}_1,\boldsymbol{p}_2)$ and $u_0(\boldsymbol{p})$ will then allow the determination of the two-particle correlation function $C(\boldsymbol{p},\boldsymbol{q})$.

The correlation function $C(\boldsymbol{p},\boldsymbol{q})$ in Eq. (15) possesses the proper coherent and chaotic limits. For a nearly completely coherent source with almost all particles populating the ground condensate state, $N_0 \to N$, the two terms in the numerator cancel each other and we have $C(\boldsymbol{p},\boldsymbol{q}) = 1$, with the absence of the BE correlation. For the other extreme of a completely chaotic source with $N_0 \ll N$, the second term in the numerator proportional to N_0^2 in Eq. (15) gives negligible contribution and can be neglected. The correlation function $C(\boldsymbol{p},\boldsymbol{q})$ then becomes the usual BE correlation for a completely chaotic source.

6 Evaluation of the Two-Particle Momentum Correlation Function

For a given number of bosons N in a spherical harmonic oscillator, the solution of fugacity z obtained as a function of the order parameter $T/\hbar\omega$ allows us to evaluate the momentum correlation function $C(p,q)$ with Eqs. (15)-(18). In Fig. 4, we show $C(p,q)$ for example for the case of N=2000 for which the critical order parameter is $T_c/\hbar\omega$=10.97, as tabulated in Table I. We observe that the correlation function is a complicated function of the average pair momentum p and the order parameter $T/\hbar\omega$. For p=$\hbar/a$ in Fig. 4(a), the correlation function $C(p,q)$ at q=0 is close to unity for temperatures below and up to $T/\hbar\omega$=9 (below $T_c/\hbar\omega$), but increases to 2 rather abruptly at $T/\hbar\omega$=12, (above $T_c/\hbar\omega$). For p=$2\hbar/a$ in Fig. 3(b), the correlation function $C(p,q)$ at $q=0$ is substantially above unity and increases gradually as temperature increases. For p=$3\hbar/a$ in Fig. 4(c), the correlation function $C(p,q)$ at q=0 is about 2 for all cases of temperatures examined. If one follows the standard phenomenological analysis and introduces the "chaoticity" parameter λ to represent the intercept of the correlation function at zero relative momentum, then this parameter λ is a function of the average pair momentum p and temperature T

$$\lambda(p,T) = [C(p,q=0;T)-1]. \tag{19}$$

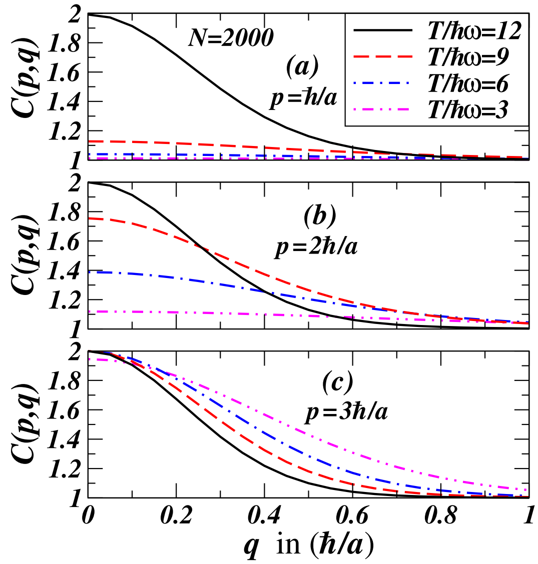

Figure 4: (Color online) The correlation function $C(\boldsymbol{p},\boldsymbol{q})$ at different values of the pair average momentum $pa/\hbar$ and temperatures. Figures (a), (b), and (c) are for p=1, 2, and $3\hbar/a$, respectively.

We display explicitly the dependence $\lambda(p,T)$ as a function of p in Fig. 5(a) for different order parameters $T/\hbar\omega$, for the case of $N = 2000$. At $T/\hbar\omega$=12, which is above the critical condensate order parameter of $T_c/\hbar\omega$=10.97, the λ parameter is 1 for all p values. At $T/\hbar\omega$=9, as p increases the λ parameter rises gradually from ∼0.1 at $p = \hbar/a$ and reaches the constant value of 1 at p=2.4$\hbar/a$. At $T/\hbar\omega$=6 and 3, for which the systems are significantly coherent with large condensate fractions, the λ parameter starts close to zero at p=$\hbar/a$, but as p increases the λ parameter increases gradually to unity at p=2.9 and 3.1$\hbar/a$ for $T/\hbar\omega$=6 and 3 respectively. The location where the λ parameter attains unity changes with temperature. The lower the temperature, the greater is the value of p at which the λ parameter attains unity.

We conclude from our results that the parameter $\lambda(p,T)$ is a sensitive function of both p and T, and $\lambda(p,T) = 1$ is not a consistent measure of the absence of the condensate fraction, as it attains the value of unity in some kinematic regions for significantly coherent systems with large condensate fractions at temperatures much below T_c. Only for the region of small p will the parameter $\lambda(p,T)$ be correlated with the chaotic fraction $f_T(T)$ of the system.

It is interesting to note that experimentally measured values of λ from different collaborations and different method of analysis [37, 38] exhibit an increase of λ as p_T of the average momentum of the pair increase as shown in Fig. 5(b). There is a similar trend of increasing λ as a function of p_T. This may be an indication of the dependence of the correlation function on the average momentum of the

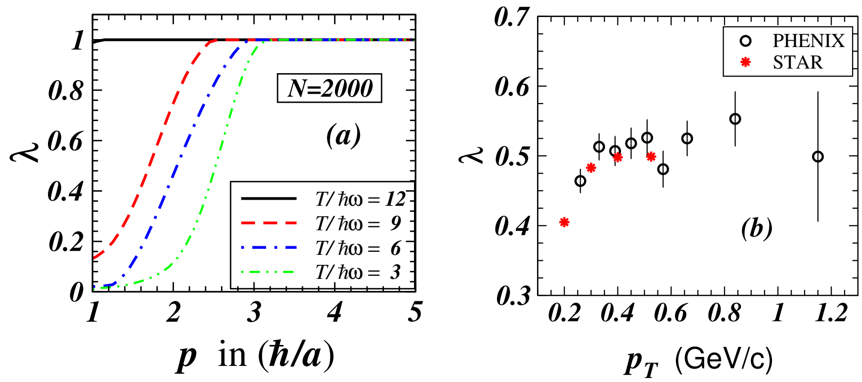

Figure 5: (Color Online) (a) The parameter λ as a function of p for different temperatures for N=2000. (b) Experimental measured values of λ as a function of p_T for AuAu Collisions at RHIC at $\sqrt{s_{\rm NN}}$=200 GeV from the PHENIX Collaboration [37] and the STAR Collaboration [38].

pair arising for a partially coherent pion source. The increase of λ as a function of the average pair momentum has also been obtained in the q-deformed harmonic oscillator model of Bose-Einstein correlations [18].

7 Bose-Einstein Condensation of Pions in their Men Fields

With regard to heavy-ion collisions at RHIC & LHC, it is instructive to raise the following question. If we have a pion system that has a root-mean squared radius $r_{\rm rms}$=10 fm, the number of identical pions N from a few hundred to a few thousand, at a freezeout temperature T=80 to 160 MeV, typical of those revealed by Bose-Einstein correlation measurements [37, 38], then, what will be the condensate fraction f_0? To answer this question, it is useful to calculate the root-mean-squared radius $r_{\rm rms}$ of the pion system as a function of the order parameter $T/\hbar\omega$ for a pion system with N=250 to 2000 as shown in Fig. 6. We can schematically represent the functional relation between $r_{\rm rms}/a$ and $T/\hbar\omega$ in Fig. 6 as

$$r_{\rm rms}/a = f_N(T/\hbar\omega). \tag{20}$$

For a given value of N and $r_{\rm rms}$, as a is equal to $\hbar/\sqrt{m_\pi\hbar\omega}$, the above equation contains only a single variable $\hbar\omega$ that can be determined as a function of T. Subsequently, the order parameter $T/\hbar\omega$ and the condensate fraction f_0 can also be determined as a function of T as shown in Fig. 7.

One finds that for the pion system with a given root-mean-squared radius of 10 fm, the value of $\hbar\omega$ ranges from about 12 to 20 MeV for N=2000 and about 20 to 30 MeV for N=250. The ratio of $T/\hbar\omega$ about 7 for N=2000, and is about 4.5 for N=250, as shown in Fig. 7(b). From these ratios of $T/\hbar\omega$, one can use Fig. 2 to

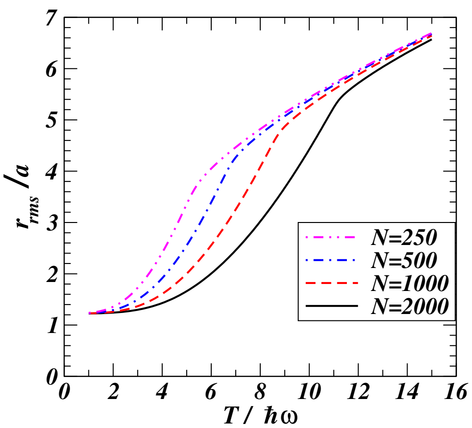

Figure 6: (Color online) The root-mean-squared radius in unit of a and the root-mean-squared momentum in units of $\hbar/a$, as a function of $T/\hbar\omega$ for different numbers of identical bosons in the system.

find out the condensate fraction. The condensate fractions $f_0(T)$ for a pion gas at various temperatures with N=2000 and N=250 are shown in Fig. 7(c). One finds that $f_0(T)$ is about $0.67 - 0.8$ for N=2000 and is about 0.9 for N=250.

We reach the conclusion from the above study that if a non relativistic pion system maintains a static equilibrium within its mean field, and if it contains a root-mean-squared radius, a pion number, and a temperature typical of those in high-energy heavy-ion collisions, then it will contain a large fraction of the Bose-Einstein pion condensate. For a relativistic pion system, while the absolute scale of the order parameter $T/\hbar\omega$ may change, the condensate fraction f_0 remains substantial [15]. Pion condensation will affect the parameter λ in momentum correlation measurements.

8 Three-particle Correlations and Coherence

Bose-Einstein condensation has important influence on the three-particle correlation function. We can determine the dependence of the three-particle correlation function on the degree of Bose-Einstein coherence in a way similar to what has been carried out for two-particle correlations.

The extraction of the coherence properties from experimental three-particle correlation data has the advantage that the problems of the resonances can be minimized. It has however the disadvantage that the statistics in the number of three-particle events may be lowered because of the restriction on the occurrence of three-particle coincidences.

Recently there has much interest in three-particle correlation measurements [22]. Bose-Einstein condensation of pions in a heavy-ion collision may suppress Bose-Einstein correlations. Furthermore, initial conditions such as the color-glass condensate (CGC) with the coherent production of partons [20] may also lead to

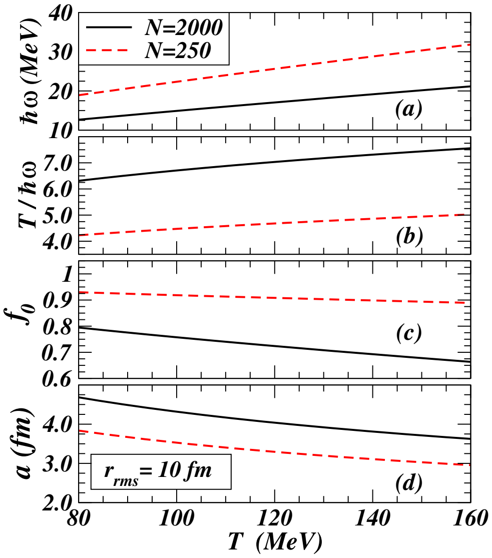

Figure 7: (Color online) (a) the potential strength $\hbar\omega$, (b) the ratio $T/\hbar\omega$, (c) the condensate fraction f_0, and (d) the oscillator length parameter a for non-relativistic boson systems with $N = 2000$ and $N = 250$ in a static equilibrium with a $r_{\rm rms} = 10$ fm, plotted as a function of the temperature T.

condensate formation [21]. Experimental results indicate the presence of a substantial condensate fraction [22]. It is of interest to formulate an analytical model to investigate how the three-particle correlation function will depend on the coherence of the underlying boson system. In a completely chaotic source when multi-particle

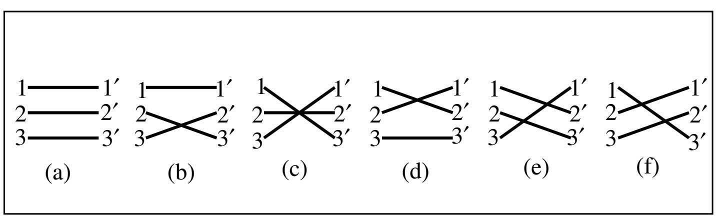

Figure 8: Three-particle distribution function expanded in terms of one-particle distribution function in uncorrelated mean-field approximation.

Bose-Einstein-type correlations are neglected, the three-particle correlation function can be written in terms of products of one-body distribution functions:

$$
\begin{aligned}
&G^{(3)}(p_1, p_2.p_3; p_1', p_2', p_3') \\
&= G^{(1)}(1,1')G^{(1)}(2,2')G^{(1)}(3,3') + G^{(1)}(1,2')G^{(1)}(2,1')G^{(1)}(3,3') \\
&\ + G^{(1)}(1,3')G^{(1)}(2,2')G^{(1)}(3,1') + G^{(1)}(1,1')G^{(1)}(2,3')G^{(1)}(3,2') \\
&\ + G^{(1)}(1,3')G^{(1)}(2,1')G^{(1)}(3,2') + G^{(1)}(1,2')G^{(1)}(2,3')G^{(1)}(3,1'), \quad (21)
\end{aligned}
$$

as represented by the diagrams in Fig. 8. With Bose-Einstein correlations, we can generalize our two-particle correlation case to the three-particle correlation functions and write down the three-particle correlation function as

$$\begin{aligned}
C(p_1,p_2,p_3) \;&\equiv \left.\frac{G^{(3)}(1,2,3;1',2',3')}{G^{(1)}(1,1')G^{(1)}(2,2')G^{(1)}(3,3')}\right|_{1'\to 1,2'\to 2,3'\to 3} \\
&= 1+\frac{G^{(1)}(1,2)G^{(1)}(2,1)-N_0u_0^2(p_1)u_0^2(p_2)}{G^{(1)}(1,1)G^{(1)}(2,2)} \\
&\quad +\frac{G^{(1)}(1,3)G^{(1)}(3,1)-N_0^2u_0^2(p_1)u_0^2(p_3)}{G^{(1)}(1,1)G^{(1)}(3,3)} \\
&\quad +\frac{G^{(1)}(2,3)G^{(1)}(3,2)-N_0^2u_0^2(p_2)u_0^2(p_3)}{G^{(1)}(2,2)G^{(1)}(3,3)} \\
&+\frac{G^{(1)}(1,3)G^{(1)}(2,1)G^{(1)}(3,2)-N_0^3u_0^2(p_1)u_0^2(p_2)u_0^2(p_3)}{G^{(1)}(1,1)G^{(1)}(2,2)G^{(1)}(3,3)} \\
&+\frac{G^{(1)}(1,2)G^{(1)}(2,3)G^{(1)}(3,1)-N_0^3u_0^2(p_1)u_0^2(p_2)u_0^2(p_3)}{G^{(1)}(1,1)G^{(1)}(2,2)G^{(1)}(3,3)}. \quad (22)
\end{aligned}$$

The above correlation function $C(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$ possesses the proper coherent and chaotic limits. For a nearly completely coherent source with almost all particles populating the ground condensate state, $N_0\to N$, the terms in the numerator cancel each other and we have $C(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$=1, and the BE correlation is absent. For the other extreme of a completely chaotic source with $N_0 \ll N$, the second terms in the numerators proportional to N_0^2 give negligible contribution and can be neglected. The correlation function $C(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3)$ becomes the usual BE correlation for a completely chaotic source. These results will allow the evaluation of the three-particle correlation function using the functions of $G^{(1)}(\boldsymbol{p}_1,\boldsymbol{p}_2)$ and $u_0^2(\boldsymbol{p})$ in Eqs. (15)-(18). Different ways of re-combining some of the terms in Eq. (22) in terms of two-particle correlation functions may allow one to extract quantities that minimize the systematic errors in two-particle correlation function measurements.

9 Conclusions and Summary

A proper framework to study Bose-Einstein correlations is the theory of Bose-Einstein condensation. We examine the condition for the occurrence of the Bose-Einstein condensation in an exactly solvable model. We place identical bosons in a spherical harmonic oscillator potential that arises either externally or approximately from its own mean fields. The order parameter is $T/\hbar\omega$, the ratio of the temperature to the energy gap between the lowest and the first excited single-particle state. The degree of chaoticity or condensation is quantified by the condensate fraction $f_0 = N_0/N$ which specifies the transition from a chaotic state to a coherent condensate state. The condensate fraction f_0 is a cubic function of the order parameter $T/\hbar\omega$. The critical order parameter $T_c/\hbar\omega$ varies with the boson number N as $T_c/\hbar\omega$=$0.6777N^{0.3666}$. The transition from the completely chaotic state with

f_0=0 to the completely coherent state with $f_0 \to 1$ is a second-order-type transition under a gradual decrease of the order parameter $T/\hbar\omega$. It is not a first-order phase transition. A pion gas with $r_{\rm rms}, T$, and N, typical of those in RHIC and LHC, is expected to contain a large condensate fraction and a high degree of suppression of Bose-Einstein correlation.

The evaluation of the two-particle correlation function indicates that the usual "chaoticity parameter" λ can only be interpreted as an experimental tool to label the intercept of the correlation function $C(\boldsymbol{p},\boldsymbol{q})$ at $\boldsymbol{q}$=0. The parameter λ is correlated with the degree of chaoticity only for small values of p but is at variance from such an interpretation of chaoticity at high values of p, as shown in Figs. 4 and 5(a).

We have written out the functional form of the three-particle distribution function as a function of the momenta of the three particles that contains the proper chaotic and coherent limits. It permits the description for the transition from the chaotic states to coherent states. These results will allow the evaluation of the three-particle correlation function in an exactly solvable problem that will assist the comparison with three-particle correlation measurements.

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